Math 270 Day 7 Part 2

Chapter 3: Applications of Linear Differential Equations

Section 3.2: Compartmental Analysis

Section 3.2: Compartmental Analysis

What we'll go over in this section

- What is Compartmental Analysis?
- Mixing Problems
- Population Models
- Radioactive Decay

Section 3.2: Compartmental Analysis

What is Compartmental Analysis?

- Many applications of differential equations involve something going in and coming out of a container
- They call the container a compartment
- Sometimes there is more than 1 compartment
- The differential equation that models a one-compartment problem is…

$$\frac{dx}{dt} = input\ rate\ - output\ rate$$ discuss

- All problems in this section are one-compartment problems

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Mixing Problems

t = time
$x(t)$ = amount of substance in a tank at time t

$$\text{DE:} \qquad \frac{dx}{dt} = \text{rate in} - \text{rate out}$$

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Mixing Problems

Example 1 Consider a large tank holding 1000 L of pure water into which a brine solution of salt begins to flow at a constant rate of 6 L/min. The solution inside the tank is kept well stirred and is flowing out of the tank at a rate of 6 L/min. If the concentration of salt in the brine entering the tank is 0.1 kg/L, determine when the concentration of salt in the tank will reach 0.05 kg/L.

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Mixing Problems

Example 2 For the mixing problem described in Example 1, assume now that the brine leaves the tank at a rate of 5 L/min instead of 6 L/min, with all else being the same. Determine the concentration of salt in the tank as a function of time.

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<u>Population Models</u>: **Malthusian/Exponential Law for Population Growth**

t = time
$p(t)$ = population at time t

Assumptions:
1) Unlimited resources (food, land, etc.)
2) Death only due to natural causes
3) Birth rate and death rate proportional to the current population

<u>Malthusian/Exponential Law for Population Growth</u>

IVP: $\frac{dp}{dt} = \text{kp}$, $p(0) = p_0$, where k is a constant (discuss)

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Population Models: **Malthusian/Exponential Law for Population Growth**

Solve the Malthusian/Exponential Law for Population Growth

IVP: $\frac{dp}{dt} = \text{kp}$, $p(0) = p_0$, where k is a constant

Solution: $p(t) = p_0 e^{kt}$

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Population Models: **Malthusian/Exponential Law for Population Growth**

Example 3 In 1790 the population of the United States was 3.93 million, and in 1890 it was 62.98 million. Using the Malthusian model, estimate the U.S. population as a function of time.

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Population Models

TABLE 3.1 A Comparison of the Malthusian and Logistic Models with U.S. Census Data (Population is given in Millions)

Year	U.S. Census	Malthusian (Example 3)	Logistic (Example 4)	$\frac{1}{p}\frac{dp}{dt}$	Logistic (Least Squares)
1790	3.93	3.93	3.93		4.11
1800	5.31	5.19	5.30	0.0312	5.42
1810	7.24	6.84	7.13	0.0299	7.14
1820	9.64	9.03	9.58	0.0292	9.39
1830	12.87	11.92	12.82	0.0289	12.33
1840	17.07	15.73	17.07	0.0302	16.14
1850	23.19	20.76	22.60	0.0310	21.05
1860	31.44	27.40	29.70	0.0265	27.33
1870	39.82	36.16	38.66	0.0235	35.28
1880	50.19	47.72	49.71	0.0231	45.21
1890	62.98	62.98	62.98	0.0207	57.41
1900	76.21	83.12	78.42	0.0192	72.11
1910	92.23	109.69	95.73	0.0162	89.37
1920	106.02	144.76	114.34	0.0146	109.10
1930	123.20	191.05	133.48	0.0106	130.92
1940	132.16	252.13	152.26	0.0106	154.20
1950	151.33	333.74	169.90	0.0156	178.12
1960	179.32	439.12	185.76	0.0145	201.75
1970	203.30	579.52	199.50	0.0116	224.21
1980	226.54	764.80	211.00	0.0100	244.79
1990	248.71	1009.33	220.38	0.0110	263.01
2000	281.42	1332.03	227.84	0.0107	278.68
2010	308.75	1757.91	233.68		291.80
2020	?	2319.95	238.17		302.56

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Population Models: **Logistic Model**

t = time
$p(t)$ = population at time t

Assumptions:
1) Unlimited resources (food, land, etc.)
2) Deaths natural causes and premature deaths from malnutrition, inadequate medical supplies, communicable diseases, violent crimes, etc.
3) Birth rate and death rate proportional to the current population
4) Death rate from natural causes proportional to the current population
5) Death rate from other causes of death proportional to number of 2-party interactions $\frac{p(p-1)}{2}$

Logistic Model for Population Growth

IVP: $\frac{dp}{dt} = -\mathrm{Ap}(\mathrm{p} - \mathrm{p}_1)$, $p(0) = p_0$, where A and p_1 are constants (discuss)

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Population Models: **Logistic Model**

Solve the Logistic Model for Population Growth

IVP: $\frac{dp}{dt} = -\mathrm{Ap}(\mathrm{p} - \mathrm{p}_1)$, $p(0) = p_0$, where A and p_1 are constants

Solution: $p(t) = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}}$

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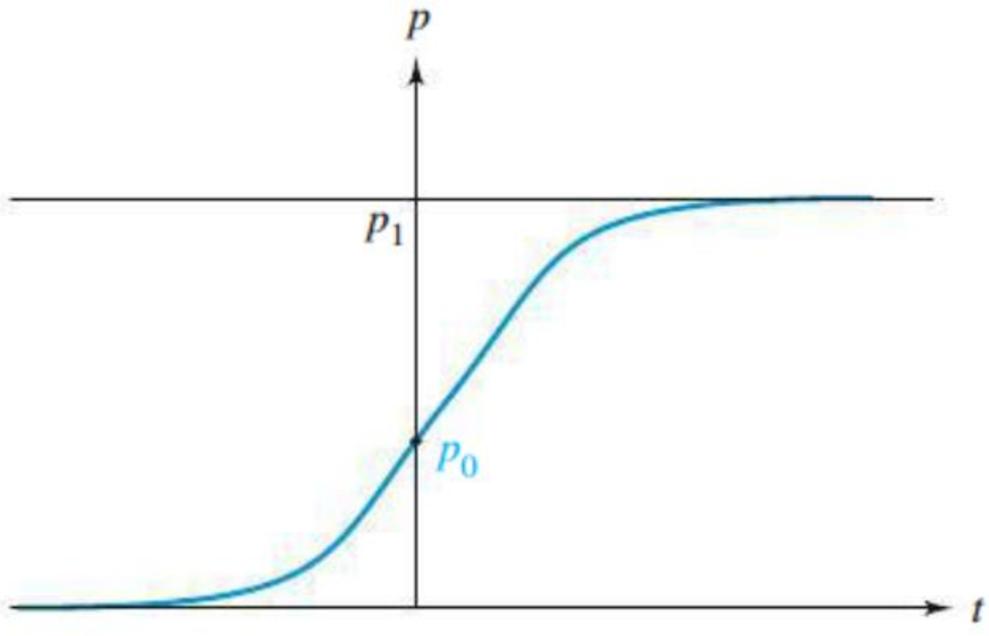
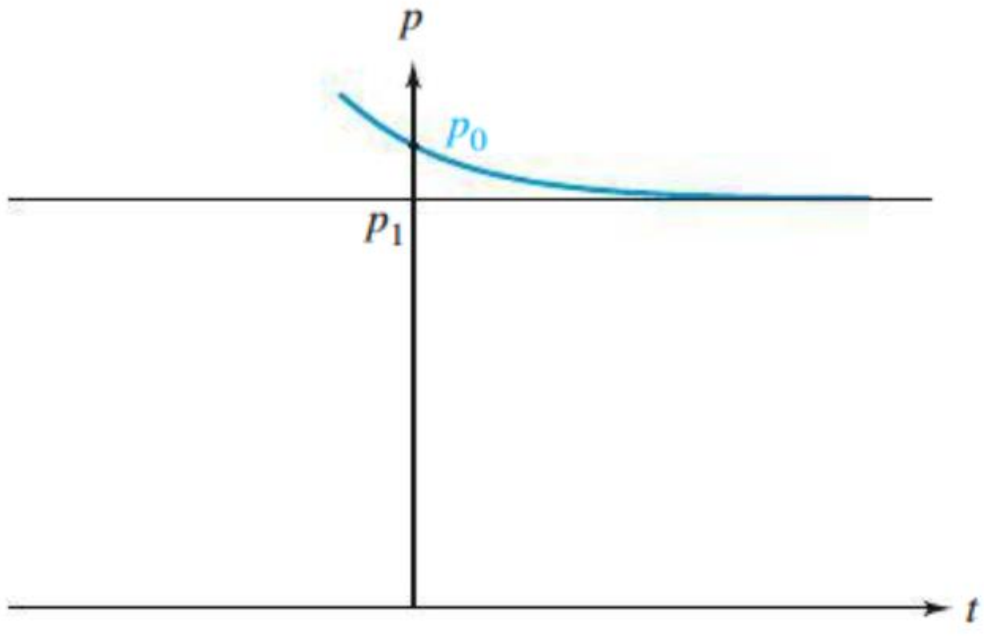
Population Models: **Logistic Model**

Solution: $p(t) = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}}$



(a) $0 < p_0 < p_1$



(b) $p_0 > p_1$

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Population Models: **Logistic Model**

Example 4 Taking the 1790 population of 3.93 million as the initial population and given the 1840 and 1890 populations of 17.07 and 62.98 million, respectively, use the logistic model to estimate the population at time t.

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<u>Radioactive Decay</u>

t = time
$x(t)$ = amount (mass or weight) of radioactive material remaining at time t

IVP: $\frac{dp}{dt} = \mathrm{kp}$, $p(0) = p_0$, where k is a constant (discuss)

<u>Solution</u>: $p(t) = p_0 e^{kt}$

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Radioactive Decay

Ex 5: A radioactive substance has a half-life of 7 years. If initially there are 300g of a radioactive substance,…

a) Find a formula for the amount of the radioactive substance remaining after t years
b) How much of the radioactive substance will remain after 12 years?
c) When will only 10g of the radioactive substance remain?